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REVIEW

The response of nonlinear controlled system under an external excitation via time delay state feedback

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Abstract An analysis of primary, superharmonic of order five, and subharmonic of order one-three resonances for non-linear s.d.o.f. system with two distinct time-delays under an external excitation is investigated. The method of multiple scales is used to determine two first order ordinary differential equations which describe the modulation of the amplitudes and the phases. Steady-state solutions and their stabilities in each resonance are studied. Numerical results are obtained by using the Software of Mathematica, which presented in a group of figures. The effect of the feedback gains and time-delays on the non-linear response of the system is discussed and it is found that: an appropriate feedback can enhance the control performance. A suitable choice of the feedback gains and time-delays can enlarge the critical force amplitude, and reduce the peak amplitude of the response (or peak amplitude of the free oscillation term) for the case of primary resonance (superharmonic resonance). Furthermore, a proper feedback can eliminate saddle-node bifurcation, thereby eliminating jump and hysteresis phenomena taking place in the corresponding uncontrolled system. For subharmonic resonance, an adequate feedback can reduce the regions of subharmonic resonance response.

Contents

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1. Introduction

It is well known that changes in stability of response, undesirable bifurcations, high-amplitude vibrations, quasiperiodic motion, and chaotic behavior may occur and cause degradation or catastrophic failure of engineering structures, for example, buildings, bridges, and airplanes subjected to vibrations caused by wind, rotating engines, and cars, or other environmental disturbances. The task of suppressing the dangerous vibrations is very important for engineering science, and bifurcation control theory has received a great deal of attention in the last years and various papers have been dedicated to the control of resonantly forced systems in various engineering fields (Chen et al., 2000).

Time-delays, which are especially prevalent if a digital control system is being implemented, can limit the performance of the feedback controllers in practical mechanical or structural systems. In many cases, unavoidable time delays in controllers and actuators give rise to complicated dynamics and can produce instability of the controlled systems. On the other hand, the time-delays can deliberately be implemented to achieve better system behavior when control is applied with the time-delays (Aernouts et al., 2000). The effect of the feedback gains and time-delays on the dynamical behavior of the controlled system is thus required to be investigated for the design of optimal controllers. The non-linear system with the time-delays has been an active topic of research over the past decades (Atay, 1998; Plaut and Hsieh, 1987a,b; Moiola et al., 1996; Hy and Zh, 2000; Hu et al., 1998; Xu and Chung, 2003; Li et al., 2006).

Atay (1998) studied the effect of delayed position feedback on the response of a van der Pol oscillator. Plaut and Hsieh (1987a) numerically analyzed the steady response of a non-linear one-degree-of-freedom mechanism with the time-delays for various sets of parameters, by a Runge–Kutta numerical integration procedure. It was found that the response might be periodic, chaotic or unbounded. By the method of multiple scales, the same authors Plaut and Hsieh (1987b) studied the effect of a damping time-delay on non-linear structural vibrations and analyzed six resonance conditions. They gave the results in a number of figures for the steady state response amplitude versus the excitation frequency and amplitude. Moiola et al. (1996) considered Hopf bifurcations in nonlinear feedback systems with time delay by using the frequency-domain approach. Two simple examples of non-linear autonomous delayed systems were presented. The computation of the two periodic branches near a degenerate Hopf bifurcation point was given. Hy and Zh (2000) considered controlled mechanical systems with time delays and, in particular, primary resonance and subharmonic resonance of a harmonically forced Duffing oscillator with time delay (stabilization of periodic motion and applications to active chassis of ground vehicles were discussed). Hu et al. (1998) considered primary resonance and 1/3 sub-harmonic resonance of a forced Duffing oscillator with time-delay state feedback. Using the multiple scales method, they demonstrated that appropriate choices of the feedback gains and the time delay are possible for better vibration control. Xu and Chung (2003) discussed a Duffing–van der Pol oscillator with time-delayed position feedback and found two routes to chaos (period-doubling bifurcation and torus breaking). Li et al. (2006) considered the response of a Duffing–van der Pol oscillator under delayed feedback control and found that unwanted multiple solutions can be prevented. It is also shown that coupled nonlinear state feedback control can be replaced by uncoupled nonlinear state feedback control. Maccari (2008) investigated the periodic solutions for parametrically excited system under state feedback control with a time delay. Using the asymptotic perturbation method, two slow-flow equations for the amplitude and phase of the parametric resonance response are derived. It is demonstrated that, if the vibration control terms are added, stable periodic solutions with arbitrarily chosen amplitude and phase can be accomplished. Therefore, an effective vibration control is possible if appropriate time delay and feedback gains are chosen. In recent work, a new time-delayed feedback control method for nonlinear oscillators has been proposed. The method has been used to suppress high-amplitude response and two-period quasiperiodic motion of a parametrically or externally excited van der Pol oscillator (Maccari, 2001, 2005). In particular, it has been shown that vibration control and quasiperiodic motion suppression are possible for appropriate choices of time delay and feedback gains. The method has also been applied to the primary resonance of a cantilever beam (Maccari, 2003) and to direct and parametric excitation of a nonlinear cantilever beam of varying orientation (Yaman, 2009). El-Gohary and El-Ganaini (2012) considered how to control the dynamic system behavior represented by a beam at simultaneous primary and sub-harmonic resonance condition, where the system damage is probable. Control is conducted via time delay absorber to suppress chaotic vibrations. A comprehensive investigation of the effect of the time delay on the control of a beam when subjected to multi-parametric excitation forces is presented.

The main objective in this paper is to study the non-linear dynamical behavior of a harmonically excited non-linear single-degree-of-freedom (s.d.o.f.) system and its control by the appropriate choice of feedback gains and two distinct time-delays under primary, super-harmonic of order five and sub-harmonic resonance of order 1/3. Two time-delays are proposed in the proportional and derivative feedback. The governing equation of motion is assumed in the following form:

$$\ddot{x} + \omega_0^2 x + \varepsilon(2\mu\dot{x} + \alpha_1 x^2 + \alpha_2 x^3 + \alpha_3 x^4 + \alpha_4 x^5) = K\cos(\Omega t) + u(x, \dot{x}) \tag{1}$$

where,

$$u(x, \dot{x}) = 2\varepsilon[d_m x(t - \tau_1) + d_n \dot{x}(t - \tau_2)].$$

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The dot denotes differentiation with respect to time. This system is related to the simplest model for many practical controlled systems, such as active vehicle suspension systems when the non-linearity in the tires is taken into account (Palkovics and Venhovens, 1992). In Eq. (1), we confine the study to the case of small damping, weak non-linearity and small feedback gains and all of the same order. In the remainder of this paper, the method of multiple scales (Elnaggar et al., 2011) is applied to Eq. (1) and three resonance conditions are examined. Attention here is focused on the effect of the time-delays and feedback gains on the steady-state response. It is believed that the result will be of value in the design of optimal controllers for this general non-linear s.d.o.f. system.

2. System analysis

2.1. Primary resonance

In this section, we examine the case of primary resonance where the excitation amplitude and frequency are introduced as

$$K = 2\varepsilon f, \quad \omega = \omega + \varepsilon\sigma. \tag{2}$$

where σ is detuning parameter. Using the method of multiple scales (Elnaggar et al., 2011), one assumes an approximate solution of Eq. (1) in the form

$$x(t;\varepsilon) = x_0(T_0, T_1) + \varepsilon x_1(T_0, T_1) + \ldots\ldots\ldots, \tag{3}$$

where $T_n = \varepsilon^n t$, $n = 0, 1, 2, \ldots\ldots$ Substituting Eq. (3) into Eq. (1) and equating the coefficients of like powers, one has the following equations to order $O(1)$ and to order $O(\varepsilon)$:

$$\begin{aligned} O(1) &: D_0^2 x_0 + \omega_0^2 x_0 = 0, \\ O(\varepsilon) &: D_0^2 x_1 + \omega_0^2 x_1 = -2D_0 D_1 x_0 - 2\mu D_0 x_0 - \alpha_1 x_0^2 \\ &\qquad - \alpha_2 x_0^3 - \alpha_3 x_0^4 - \alpha_4 x_0^5 + 2f\cos(\omega t) \\ &\qquad + 2\varepsilon[d_m x(t-\tau_1) + d_n \dot{x}(t-\tau_2)]. \end{aligned} \tag{4}$$

where $D_n = \partial/\partial T_n$. Solving the first equation in (4) for $x_0(T_0, T_1)$, we have

$$x_0(T_0, T_1) = A(T_1)e^{i\omega_0 T_0} + \bar{A}(T_1)e^{-i\omega_0 T_0} \tag{5}$$

where $A(T_1)$ is a complex-valued quantity that will be determined by imposing the solvability condition. Substituting Eq. (5) into the second equation of (4) and eliminating the secular terms, one obtains

$$\begin{aligned} &-2i\omega_0 A' - 2i\mu\omega_0 A - 3\alpha_2 A^2\bar{A} - 10\alpha_4 A^3\bar{A}^2 + 2d_m A e^{-i\omega_0\tau_1} \\ &\quad + 2id_n\omega_0 A e^{-i\omega_0\tau_2} + f e^{-i\varepsilon\sigma T_0} = 0 \end{aligned} \tag{6}$$

A first order approximate solution of Eq. (1) can be written as

$$x = a\cos(\Omega t - \gamma) + O(\varepsilon). \tag{7}$$

where $\gamma = \sigma T_1 - \beta(T_1)$. The amplitude a and phase γ of the response are governed by the following polar form of modulation equations:

$$\begin{aligned} a' &= -\left(\mu + \frac{d_m}{\omega_0}\sin(\omega_0\tau_1) - d_n\cos(\omega_0\tau_2)\right)a + \frac{f}{\omega_0}\sin(\gamma), \\ a\gamma' &= \left(\sigma + \frac{d_m}{\omega_0}\cos(\omega_0\tau_1) + d_n\sin(\omega_0\tau_2)\right)a - \frac{3\alpha_2}{8\omega_0}a^3 - \frac{5\alpha_4}{16\omega_0}a^5 \\ &\quad + \frac{f}{\omega_0}\cos(\gamma). \end{aligned} \tag{8}$$

where a prime indicates the derivative with respect to the time scale T_1. Obviously, the presence of the feedback gains and time-delays modifies the averaged equations by adding two terms that are relevant to feedback control. Thus, it is possible to achieve the desirable behavior if the feedback is deliberately implemented.

From Eq. (8), we have a set of algebraic equations for amplitude a and phase γ of the steady-state solutions of Eq. (1) for the primary resonance which can be obtained by setting $a' = \gamma' = 0$. That is,

$$\begin{aligned} &-\left(\mu + \frac{d_m}{\omega_0}\sin(\omega_0\tau_1) - d_n\cos(\omega_0\tau_2)\right)a + \frac{f}{\omega_0}\sin(\gamma) \\ &\quad = 0, \left(\sigma + \frac{d_m}{\omega_0}\cos(\omega_0\tau_1) + d_n\sin(\omega_0\tau_2)\right)a - \frac{3\alpha_2}{8\omega_0}a^3 \\ &\quad - \frac{5\alpha_4}{16\omega_0}a^5 + \frac{f}{\omega_0}\cos(\gamma) = 0. \end{aligned} \tag{9}$$

From the system (9), whereby we derive the frequency–response relation (bifurcation equation) between a and σ:

$$\left[\mu_0^2 + \left(\sigma_0 - \frac{3\alpha_2}{8\omega_0}a^2 - \frac{5\alpha_4}{16\omega_0}a^4\right)^2\right]a^2 = \left(\frac{f}{\omega_0}\right)^2, \tag{10}$$

where

$$\mu_0 = \mu + \frac{d_m}{\omega_0}\sin(\omega_0\tau_1) - d_n\cos(\omega_0\tau_2),$$

$$\sigma_0 = \sigma + \frac{d_m}{\omega_0}\cos(\omega_0\tau_1) + d_n\sin(\omega_0\tau_2)$$

The amplitude of the response a is a function of the external detuning σ, feedback gains, Time-delays and the amplitude of the excitation f. The peak amplitude a_p of the primary resonance response, obtained from Eq. (10), is given by

$$a_p = \frac{f}{\omega_0\mu_0}. \tag{11}$$

The real solution a of Eq. (10) determines the primary resonance response amplitude. There can be either one or three real solutions. Three real solutions exist between two points of vertical tangents (saddle-node bifurcation) (Nayfeh and Mook, 1979; Nayfeh and Balachandran, 1995), which are determined by differentiation of Eq. (10) implicitly with respect to a^2. This leads to the condition

$$\begin{aligned} &\sigma_0^2 - \left(\frac{3\alpha_2}{2\omega_0}a^2 + \frac{15\alpha_4}{8\omega_0}a^4\right)\sigma_0 + \frac{27\alpha_2^2}{64\omega_0^2}a^4 + \frac{15\alpha_2\alpha_4}{16\omega_0^2}a^6 \\ &\quad + \frac{125\alpha_4^2}{256\omega_0^2}a^8 + \mu_0^2 = 0 \end{aligned} \tag{12}$$

with solutions

$$\begin{aligned} \sigma_0^{\pm} &= \left(\frac{3\alpha_2}{4\omega_0}a^2 + \frac{15\alpha_4}{16\omega_0}a^4\right) \\ &\quad \pm \left[\left(\frac{3\alpha_2}{8\omega_0}a^2 + \frac{5\alpha_4}{8\omega_0^2}a^4\right)^2 - \mu_0^2\right]^{1/2} \end{aligned} \tag{13}$$

For $\left(\frac{3\alpha_2}{8\omega_0}a^2 + \frac{5\alpha_4}{8\omega_0^2}a^4\right) \succ \mu_0$, there exists an interval $\sigma_0^- \prec \sigma_0 \prec \sigma_0^+$ in which three real and positive solutions a of Eq. (10) exist. In the limit $\left(\frac{3\alpha_2}{8\omega_0}a^2 + \frac{5\alpha_4}{8\omega_0^2}a^4\right) \to \mu_0$, this interval shrinks to the point $\sigma_0 = \frac{3\alpha_2}{4\omega_0}a^2 + \frac{15\alpha_4}{16\omega_0}a^4$. The critical force amplitude obtained from Eq. (10) is

$$f_{crit} = \mu_0\omega_0\left[-\frac{6\alpha_2}{10\alpha_4} + \left(\frac{36\alpha_2^2}{100\alpha_4} + \frac{32\omega_0\mu_0}{5\alpha_4}\right)^{1/2}\right]^{1/2} \tag{14}$$

For $f \prec f_{crit}$ there is only one solution while for $f\ f_{crit}$ there are three. The stability of the solutions is determined by the eigenvalues of the corresponding Jacobian matrix of Eq. (8). The corresponding eigenvalues are the roots of

$$\lambda^2 + 2\mu_0\lambda + \Gamma = 0. \tag{15}$$

where

$$\Gamma = \mu_0^2 + \sigma_0^2 + \frac{27\alpha_2^2}{64\omega_0^2}a^4 + \frac{15\alpha_2\alpha_4}{16\omega_0^2}a^6 + \frac{125\alpha_4^2}{256\omega_0^2}a^8 - \frac{3\alpha_2\sigma_0}{2\omega_0}a^2 - \frac{15\alpha_4\sigma_0}{8\omega_0}a^4$$

From Eq. (15), it is found that the sum of the two eigenvalues is $-2\mu_0$ but for the uncontrolled system, the sum of the two eigenvalues is -2μ, which is negative (Nayfeh and Mook, 1979; Nayfeh and Balachandran, 1995). The addition of the feedback gains and time-delays modifies the sum of the two eigenvalues. Then we have three cases such as $\mu_0\ 0.0$, $\mu_0 = 0.0$ and $\mu_0 \prec 0.0$ that may occur depending on the values of the feedback gains and time-delays. If the feedback gains and time-delays are chosen in such a way that the sum of the two eigenvalues is positive ($\mu_0 \prec 0.0$), at least one of the two eigenvalues will always have a positive real part. The system will be unstable. The selection of the feedback gains and time-delays is not possible. On the other hand, if the sum of the two eigenvalues is zero ($\mu_0 = 0.0$) by a certain value of the feedback gains and time-delays, a pair of purely imaginary eigenvalues and hence a Hopf bifurcation (i.e. when $\mu_0 = 0.0$ and $\Gamma\ 0$ are satisfied together) may occur. Therefore, the above two cases should be avoided from the viewpoint of bifurcation control. The feedback should be implemented at least in such a way that $\mu_0\ 0.0$ is satisfied. Under such feedback gains and time-delays, the sum of the two eigenvalues is always negative, and accordingly, at least one of the two eigenvalues will always have a negative real part. The other eigenvalue is zero when $\Gamma = 0$, where a saddle-node bifurcation occurs.

It has been shown that the feedback gains and time-delays can change the quantities of μ_0 and σ_0, which govern the critical force amplitude, the peak amplitude of the primary resonance response and the stability of steady state motions. The peak amplitude of the response a_{p} is inversely proportional to μ_0. Thus, the peak amplitude of the response a_{p} decreases (or increases) as μ_0 increases (or decreases). On the other hand, if the resulting μ_0 and σ_0 maintain the inequality $\Gamma\ 0$, there is no unstable solution. The system will not exhibit jump and hysteresis phenomenon. Thus, the appropriate feedback gains and time-delays can improve the control performance.

2.2. Super-harmonic resonance

To analyze the super-harmonic resonance, the amplitude and frequency of excitation are expressed as

$$K = 2f, \quad 5\Omega = \omega + \varepsilon\sigma. \tag{16}$$

Using the method of multiple scales, one obtains the first order approximation for the super-harmonic resonance response

$$x = a\cos(5\Omega t - \gamma) + 2f(\omega_0^2 - \Omega^2)^{-1}\cos(\Omega t) + o(\varepsilon), \tag{17}$$

where $\gamma = \sigma T_1 - \beta(T_1)$. The amplitude a and phase γ of the free oscillation term are governed by

$$a' = -\mu_0 a - \frac{\alpha_4\Lambda^5}{\omega_0}\sin(\gamma),$$
$$a\gamma' = \left(\sigma_0 - \frac{3\alpha_2\Lambda^2}{\omega_0} - \frac{15\alpha_4\Lambda^4}{\omega_0}\right)a - \frac{3\alpha_2}{8\omega_0}a^3 - \frac{15\alpha_4\Lambda^2}{2\omega_0}a^3 - \frac{5\alpha_4}{16\omega_0}a^5 - \frac{\alpha_4\Lambda^5}{\omega_0}\cos(\gamma). \tag{18}$$

where $\Lambda = f(\omega_0^2 - \Omega^2)^{-1}$.

The fixed points of this system are given by

$$-\mu_0 a - \frac{\alpha_4\Lambda^5}{\omega_0}\sin(\gamma) = 0,\ \left(\sigma_0 - \frac{3\alpha_2\Lambda^2}{\omega_0} - \frac{15\alpha_4\Lambda^4}{\omega_0}\right)a - \left(\frac{3\alpha_2}{8\omega_0} + \frac{15\alpha_4\Lambda^2}{2\omega_0}\right)a^3 - \frac{5\alpha_4}{16\omega_0}a^5 - \frac{\alpha_4\Lambda^5}{\omega_0}\cos(\gamma) = 0. \tag{19}$$

Squaring and adding these equations, one has the frequency–response equation

$$\left[\mu_0^2 + \left(\sigma_0 - \frac{3\alpha_2\Lambda^2}{\omega_0} - \frac{15\alpha_4\Lambda^4}{\omega_0} - \left(\frac{3\alpha_2}{8\omega_0} + \frac{15\alpha_4\Lambda^2}{2\omega_0}\right)a^2 - \frac{5\alpha_4}{16\omega_0}a^4\right)^2\right]a^2 = \frac{\alpha_4^2\Lambda^{10}}{\omega_0^2}. \tag{20}$$

The real solution a of Eq. (20) determines the superharmonic resonance response amplitude. There can be either one or three real solutions. Three real solutions exist between two points of vertical tangents (Saddle-node bifurcation), which are determined by differentiation of Eq. (20) with respect to a^2. This leads to the condition

$$\sigma_0^{\pm} = Z \pm \left[\left(\frac{3\alpha_2}{8\omega_0}a^2 + \frac{5\alpha_4}{8\omega_0}a^4 + \frac{15\alpha_4\Lambda^2}{2\omega_0}a^2\right)^2 - \mu_0^2\right]^{1/2} \tag{21}$$

where

$$Z = \frac{3\alpha_2}{4\omega_0}a^2 + \frac{3\alpha_2\Lambda^2}{\omega_0} + \frac{15\alpha_4}{16\omega_0}a^4 + \frac{15\alpha_4\Lambda^2}{\omega_0}a^2 + \frac{15\alpha_4\Lambda^4}{\omega_0}$$

For $\left(\frac{3\alpha_2}{8\omega_0} + \frac{5\alpha_4}{8\omega_0}a^2 + \frac{15\alpha_4\Lambda^2}{2\omega_0}\right)a^2 \succ \mu_0$, there exists an interval $\sigma_0^- \prec \sigma_0 \prec \sigma_0^+$ in which three real and positive solutions a of Eq. (20) exist. In the limit $\left(\frac{3\alpha_2}{8\omega_0} + \frac{5\alpha_4}{8\omega_0}a^2 + \frac{15\alpha_4\Lambda^2}{2\omega_0}\right)a^2 \to \mu_0$, this interval shrinks to the point $\sigma_0 = Z$. The critical force amplitude obtained from Eq. (20) is the solution of the following equation

$$\Lambda^{10} = \frac{2\mu_0^2\omega_0^2}{10\alpha_4^3}\left(-3\alpha_2 - 60\alpha_4\Lambda^2 + \sqrt{(3\alpha_2 + 60\alpha_4\Lambda^2)^2 + 160\alpha_4\mu_0\omega_0}\right) \tag{22}$$

where $\Lambda = \frac{f_{crit}}{\omega_0^2 - \Omega^2}$.

For $f \prec f_{crit}$ there is only one solution while for $f \succ f_{crit}$ there are three. The peak amplitude of the free oscillation term is given by

$$a_p = \frac{|\alpha_4|\Lambda^5}{\omega_0\mu_0}. \tag{23}$$

which is also inversely proportional to μ_0. Increasing μ_0 can diminish the value of a_{p}.

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The stability of the steady state super-harmonic resonance response is determined by the eigenvalues of the corresponding Jacobian matrix, which are the roots of

$$\lambda^2 + 2\mu_0\lambda + \Gamma_1 = 0. \tag{24}$$

where

$$\begin{aligned}\Gamma_1 &= \mu_0^2 + \sigma_0^2 + \frac{27\alpha_2^2}{64\omega_0^2}a^4 + \frac{9\alpha_2^2\Lambda^2}{2\omega_0^2}a^2 + \frac{15\alpha_2\alpha_4}{16\omega_0^2}a^6 + \frac{125\alpha_4^2}{256\omega_0^2}a^8 \\ &\quad - \frac{3\alpha_2\sigma_0}{2\omega_0}a^2 - \frac{15\alpha_4\sigma_0}{8\omega_0}a^4 \quad + \frac{9\alpha_2^2\Lambda^4}{\omega_0^2} + \frac{45\alpha_2\alpha_4\Lambda^2}{2\omega_0^2}a^4 \\ &\quad + \frac{225\alpha_2\alpha_4\Lambda^4}{2\omega_0^2}a^2 + \frac{90\alpha_2\alpha_4\Lambda^6}{\omega_0^2} + \frac{75\alpha_4^2\Lambda^2}{4\omega_0^2}a^6 + \frac{225\alpha_4^2\Lambda^8}{\omega_0^2} \\ &\quad + \frac{1575\alpha_4^2\Lambda^4}{8\omega_0^2}a^4 + \frac{450\alpha_4^2\Lambda^6}{\omega_0^2}a^2 - \frac{30\alpha_4\sigma_0\Lambda^2}{\omega_0}a^2 - \frac{6\alpha_2\sigma_0\Lambda^2}{\omega_0} \\ &\quad - \frac{30\alpha_4\sigma_0\Lambda^4}{\omega_0}.\end{aligned}$$

The steady state motions are stable only when the two inequalities μ_0 0 and Γ_1 0 simultaneously hold, and are otherwise unstable. Against the second inequality would imply the existence of an eigenvalue having a positive real part. Replacing the second inequality by equality yields the critical parameters corresponding to saddle-node bifurcation. The suitable choice of the feedback gains and time-delays can improve the control performance. Moreover, the occurrence of saddle-node bifurcation, the jump and hysteresis phenomena can be delayed or eliminated.

2.3. Subharmonic resonance

Periodically forced, nonlinear oscillators often show a strong response when driven near a rational multiple of the natural frequency of the linearized system: Such occurrences are known as subharmonic resonances. In these systems, at these resonances, nonlinearities conspire to shift the response from the driving frequency to (near) the natural frequency of the linearized system. In the case of subharmonic resonance, it is assumed that K = 2f and $\Omega = 3\omega + \varepsilon\sigma$. The first order approximation for the steady state subharmonic resonance response is given by

$$x = a\cos\left(\frac{1}{3}(\Omega t - \gamma)\right) + 2f(\omega_0^2 - \Omega^2)^{-1}\cos(\Omega t) + O(\varepsilon), \tag{25}$$

where $\gamma = \sigma T_1 - 3\beta(T_1)$. The amplitude a and phase γ are governed by the following polar form of modulation equations

$$\begin{aligned}a' &= -\mu_0 a - \frac{3\alpha_2\Lambda}{4\omega_0}a^2\sin(\gamma) - \frac{15\alpha_4\Lambda^3}{2\omega_0}a^2\sin(\gamma) - \frac{5\alpha_4\Lambda}{4\omega_0}a^4\sin(\gamma), \\ a\gamma' &= \left(\sigma_0 - \frac{9\alpha_2\Lambda^2}{\omega_0} - \frac{45\alpha_4\Lambda^4}{\omega_0}\right)a - \frac{9\alpha_2}{8\omega_0}a^3 - \frac{45\alpha_4\Lambda^2}{2\omega_0}a^3 \\ &\quad - \frac{15\alpha_4}{16\omega_0}a^5 - \frac{9\alpha_2\Lambda}{4\omega_0}a^2\cos(\gamma). - \frac{45\alpha_4\Lambda^3}{2\omega_0}a^2\cos(\gamma) \\ &\quad - \frac{15\alpha_4\Lambda}{4\omega_0}a^4\cos(\gamma)\end{aligned} \tag{26}$$

where $\Lambda = f(\omega_0^2 - \Omega^2)^{-1}$. The steady state response corresponds to the solutions of

$$\begin{aligned}&-\mu_0 a - \frac{3\alpha_2\Lambda}{4\omega_0}a^2\sin(\gamma) - \frac{15\alpha_4\Lambda^3}{2\omega_0}a^2\sin(\gamma) - \frac{5\alpha_4\Lambda}{4\omega_0}a^4\sin(\gamma) = 0, \\ &\times\left(\sigma_0 - \frac{9\alpha_2\Lambda^2}{\omega_0} - \frac{45\alpha_4\Lambda^4}{\omega_0}\right)a - \frac{9\alpha_2}{8\omega_0}a^3 - \frac{45\alpha_4\Lambda^2}{2\omega_0}a^3 - \frac{15\alpha_4}{16\omega_0}a^5 \\ &- \frac{9\alpha_2\Lambda}{4\omega_0}a^2\cos(\gamma) - \frac{45\alpha_4\Lambda^3}{2\omega_0}a^2\cos(\gamma) - \frac{15\alpha_4\Lambda}{4\omega_0}a^4\cos(\gamma) = 0.\end{aligned} \tag{27}$$

Eliminating γ from these equations, one has the frequency–response equation

$$\begin{aligned}&\left[9\mu_0^2 + \left(\sigma_0 - \frac{9\alpha_2\Lambda^2}{\omega_0} - \frac{45\alpha_4\Lambda^4}{\omega_0} - \frac{9\alpha_2}{8\omega_0}a^2 - \frac{45\alpha_4\Lambda^2}{2\omega_0}a^2 - \frac{15\alpha_4}{16\omega_0}a^4\right)^2\right]a^2 \\ &= \left[\frac{9\alpha_2\Lambda}{4\omega_0} + \frac{45\alpha_4\Lambda^3}{2\omega_0} + \frac{15\alpha_4\Lambda}{4\omega_0}a^2\right]^2 a^4.\end{aligned} \tag{28}$$

Eq. (28) shows that there are two possibilities: either a trivial solution $a = 0$, or non-trivial solutions when $a \neq 0$, which are given by

$$\begin{aligned}&\left[9\mu_0^2 + \left(\sigma_0 - \frac{9\alpha_2\Lambda^2}{\omega_0} - \frac{45\alpha_4\Lambda^4}{\omega_0} - \frac{9\alpha_2}{8\omega_0}a^2 - \frac{45\alpha_4\Lambda^2}{2\omega_0}a^2 - \frac{15\alpha_4}{16\omega_0}a^4\right)^2\right] \\ &= \left[\frac{9\alpha_2\Lambda}{4\omega_0} + \frac{45\alpha_4\Lambda^3}{2\omega_0} + \frac{15\alpha_4\Lambda}{4\omega_0}a^2\right]^2 a^2.\end{aligned} \tag{29}$$

The steady state solutions of subharmonic resonance response is determined by the eigenvalues of the characteristic equation, which are the roots of

$$\lambda^2 + \frac{6(\alpha_2 + 10\alpha_4\Lambda^2)\mu_0}{3\alpha_2 + 5\alpha_4(a^2 + 6\Lambda^2)}\lambda + \Gamma_2 = 0. \tag{30}$$

where

$$\begin{aligned}\Gamma_2 = &\frac{1}{256(3\alpha_2 + 5\alpha_4(a^2 + 6\Lambda^2))\omega_0^2}(324(a^4 - 64\Lambda^4)\alpha_2^3 \\ &+ 225(3a^8 + 16a^6\Lambda^2 - 480a^4\Lambda^4 - 4608a^2\Lambda^6 - 11520\Lambda^8)\alpha_2\alpha_4^2 \\ &+ 375(a^{10} + 18a^8\Lambda^2 - 96a^6\Lambda^4 - 576a^4\Lambda^6 \\ &- 6912a^2\Lambda^8 - 13824\Lambda^{10})\alpha_4^3 \\ &+ 1440(a^4 + 16a^2\Lambda^2 + 48\Lambda^4)\alpha_2\alpha_4\sigma_0\omega_0 \\ &+ 800(a^6 + 42a^4\Lambda^2 + 144a^2\Lambda^4 + 288\Lambda^6)\alpha_4^2\sigma_0\omega_0 \\ &- 2304\alpha_2\mu_0^2\omega_0^2 - 11520(a^2 + 2\Lambda^2)\alpha_4\mu_0^2\omega_0^2 \\ &- 256\alpha_2\sigma_0^2\omega_0^2 - 1280(a^2 + 2\Lambda^2)\alpha_4\sigma_0^2\omega_0^2 \\ &+ 36\alpha_2^2(15(a^6 + 6a^4\Lambda^2 - 192a^2\Lambda^4 - 768\Lambda^6)\alpha_4 + 128\Lambda^2\sigma_0\omega_0))\end{aligned}$$

If all of the eigenvalues have negative real parts, the corresponding steady-state solution is stable. Otherwise, it is unstable. In other words, if the two inequalities $\frac{6(\alpha_2 + 10\alpha_4\Lambda^2)\mu_0}{3\alpha_2 + 5\alpha_4(a^2 + 6\Lambda^2)} \succ 0$ and $\Gamma_2 \succ 0$ are satisfied at the same time then the corresponding steady-state solution is stable. We note that there is no jump phenomenon in this case. Also, although the frequency of the excitation is three times the natural frequency of the system, the response is quite large. It is noted that if the feedback control is appropriately implemented, the system will reduce the regions of subharmonic resonance.

Based on the foregoing discussion, it can be concluded that the appropriate choice of time-delays can improve the control performance. A certain combination of μ_0 and σ_0 can delay or

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eliminate the occurrence of saddle-node bifurcation in the primary and superharmonic resonance responses. Moreover, regions of subharmonic resonance shrink in the controlled system by a suitable feedback control. Whenever numerical simulations are performed, the values for the system parameters are chosen as follows: $\alpha_1 = 0.5$, $\alpha_2 = 1.0$, $\alpha_3 = 1.0$, $\alpha_4 = 0.05$, $\omega_0 = 5.0$, $\mu = 0.05$, $f = 0.3$, $d_m = 0.135$ and $d_n = 0.015$, unless otherwise specified. For simplicity in the remaining part of this paper, we use the time-delay τ_2 in the form $\tau_2 = \tau_1 + \delta$

Usually, μ_0 should have a larger value in order to improve the control performance. Thence after some mathematical calculations, it follows that there are two cases to be considered depending on the quantities of the feedback gains:

Case I: If the feedback is implemented in such a way that $\mu \succ |d_m/\omega_0 + d_n|$ or $\mu \succ |d_m/\omega_0 - d_n|$, the resulting μ_0 is always positive regardless of the values of the time-delays.

Case II: If the feedback gains and time-delays are chosen under the condition $\mu \prec |d_m/\omega_0 + d_n|$ or $\mu \prec |d_m/\omega_0 - d_n|$, the resulting μ_0 may be positive, zero, or negative, depending on the different values of the time-delays. In practical engineering problem, the last two cases should be absolutely prohibited. The time-delays should be carefully designed so that μ_0 is guaranteed to be always positive.

3. Numerical results and discussion

This section illustrates the effect of the feedback gains and time-delays on the non-linear dynamical behavior of the controlled system under the primary resonance response. The results will be presented in a number of figures:

In Fig. 1a, the critical force amplitude f_{crit} is plotted as a function of the time-delay τ_1 for three different fixed δ, while the peak amplitude of the primary resonance response a_p versus the time-delay τ_1 is illustrated in Fig. 1b. It is easily noted that f_{crit} and a_p vary significantly as the time-delay τ_1 increases. If, unfortunately, the time-delay is not appropriately selected, a smaller μ_0 is acquired. Subsequently, f_{crit} reaches a smaller value too, while a_p attains a larger value. This leads to a poor control performance. For the case of two distinct time-delays in feedback control, the time-delay τ_1 should be implemented in the region $\tau_1 \prec \pi/\omega_0 \approx 0.628$. Thus, a larger f_{crit} and a smaller a_p can be obtained. For three different fixed time-delays $\tau_1 = 0.0$, $\tau_1 = \pi/4\omega_0$ and $\tau_1 = \pi/2\omega_0$, the corresponding f_{crit} and a_p are plotted in Fig. 1c and d, respectively, as functions of the difference of two time-delays δ. It is easy to see that when $\tau_1 = \pi/2\omega_0$, the critical force amplitude f_{crit} is larger than any other value of the time-delay τ_1 for a fixed δ, while a_p has a smaller value. Furthermore, f_{crit} is larger while a_p is smaller in $\delta \prec \pi/\omega_0$ than in $\delta \succ \pi/\omega_0$. Thus, the difference of two time-delays δ should be implemented in the region $\delta \prec \pi/\omega_0$ for the purpose of optimal control. The frequency–response curves (bifurcation curves) for the primary resonance response are depicted in Fig. 2 for three sets of the time-delays. The response curves have an unstable portion for the time-delays $\tau_1 = 0.0$, $\delta = 0.0$ and $\tau_1 = 0.0$, $\delta = \pi/2\omega_0$, which correspond to the cases of no time-delays and of only one derivative feedback time-delay in the feedback control. In contrast, for

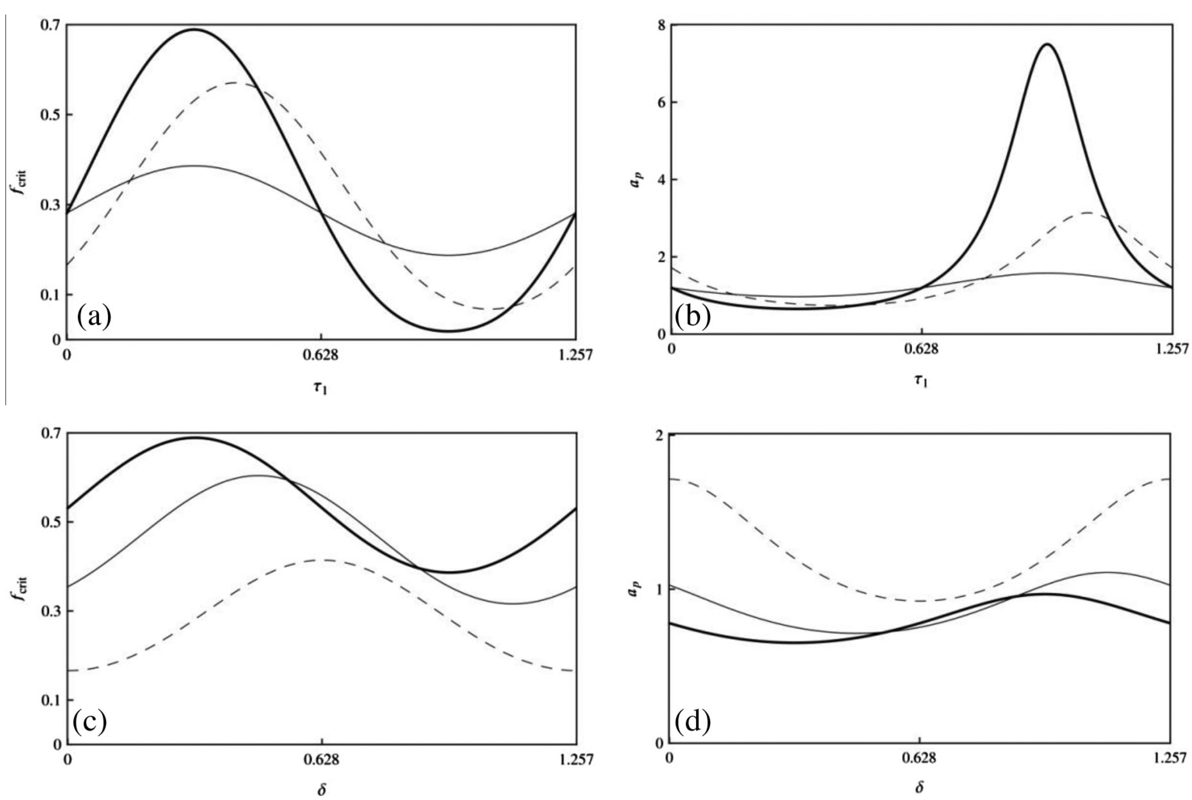

Figure 1 The critical force amplitude f_{crit} and the peak amplitude a_p of the primary resonance response as a function of the time-delays : (a) and (b) ----- curves are for $\delta = 0.0$, —— curves are for $\delta = 3\pi/2\omega_0$ and **——** curves are for $\delta = \pi/2\omega_0$; (c) and (d) ----- curves for $\tau_1 = 0.0$, —— curves for $\tau_1 = \pi/4\omega_0$ and **——** curves for $\tau_1 = \pi/2\omega_0$.

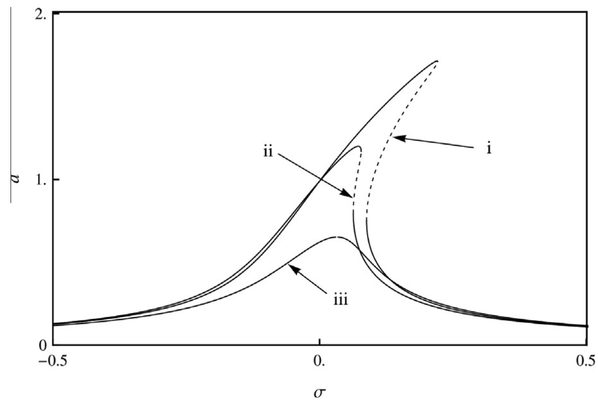

Figure 2 Frequency–response curves for primary resonance corresponding to three sets of the time-delays: (i) are curves for $\tau_1 = 0.0$ and $\delta = 0.0$, (ii) are curves for $\tau_1 = 0.0$ and $\delta = \pi/2\omega_0$ and (iii) are curves for $\tau_1 = \pi/2\omega_0$ and $\delta = \pi/2\omega_0$.

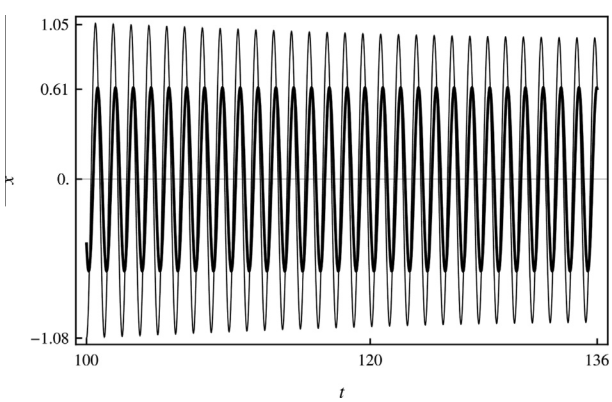

Figure 3 The time histories; thick lines for the controlled system at $d_m = 0.135$, $d_n = 0.015$, $\tau_1 = \pi/2\omega_0$ and $\tau_2 = \pi/\omega_0$, while thin lines are for the uncontrolled system.

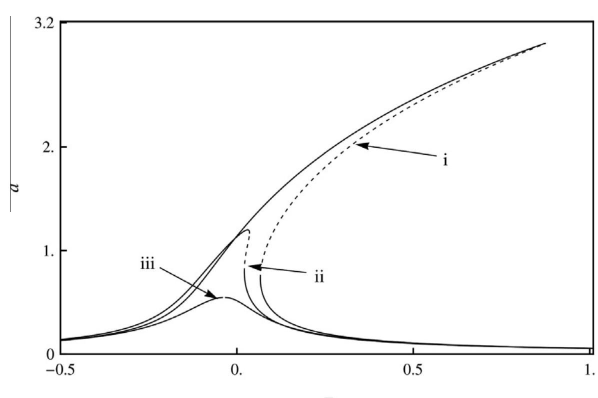

Figure 4 Frequency–response curves for primary resonance corresponding to three sets of the time-delays under $d_m = 0.27$ and $d_n = 0.03$: (i) are curves for $\tau_1 = 0.0$ and $\delta = 0.0$, (ii) are curves for$\tau_1 = 0.0$ and $\delta = \pi/2\omega_0$ and (iii) are curves for $\tau_1 = \pi/4\omega_0$ and $\delta = \pi/2\omega_0$.

two distinct time-delays $\tau_1 = \pi/2\omega_0$ and $\delta = \pi/2\omega_0$, no unstable region exists in the system response curve. This indicates that saddle-node bifurcation and jump phenomenon can be

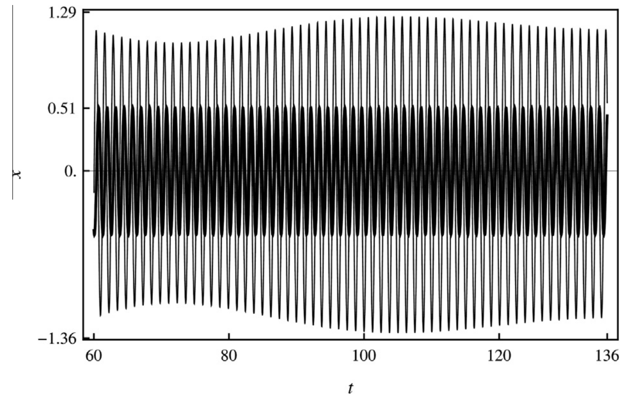

Figure 5 The time histories; thick lines for the controlled system at $d_m = 0.27$, $d_n = 0.03$, $\tau_1 = \pi/4\omega_0$ and $\tau_2 = 3\pi/4\omega_0$, while thin lines are for the uncontrolled system.

eliminated by suitable time-delays. Moreover, the peak amplitude of the primary resonance response a_p for $\tau_1 = \pi/2\omega_0$ and $\delta = \pi/2\omega_0$ is smaller than that for the other two cases. Fig. 3 shows the time histories of the uncontrolled system and controlled system (1) under the excitation amplitude$f = 0.3$, we notice that the peak amplitude is reduced.

Fig. 4 also shows the frequency–response curves for the primary resonance response with $d_m = 0.27$ and $d_n = 0.03$, under different time-delays. There exists an interval in which three solutions exist, and jump phenomenon is presented for $\tau_1 = 0.0$ and $\delta = 0.0$ and $\tau_1 = 0.0$, $\tau_1 = \pi/2\omega_0$. In contrast, for $\tau_1 = \pi/4\omega_0$, $\delta = \pi/2\omega_0$, there only exists one solution. Jump and hysteresis phenomena do not exist. This simple example also indicates that the saddle-node bifurcation and jump phenomenon can be eliminated by appropriate selection of the time-delays. Fig. 5 shows the time histories of the uncontrolled system and controlled system (1) under the excitation amplitude$f = 0.3$, $\tau_1 = \pi/4\omega_0$ and $\delta = \pi/2\omega_0$, we notice that the peak amplitude is reduced.

For the superharmonic resonance response, the suitable choice of the time-delays and feedback gains can also improve the control performance. Moreover, the occurrence of saddle-node bifurcation, jump and hysteresis phenomena can be delayed or eliminated.

As an illustration, Figs. 6a and 6b show the variation of the critical force amplitude f_{crit} and the peak amplitude of the free oscillation term a_p for $f = 29.0$ with the time- delay τ_1, under three different δ, $\delta = 0.0$, $\delta = \pi/2\omega_0$ and $\delta = 3\pi/2\omega_0$ respectively. In contrast, Figs. 6c and 6d demonstrates the variation of f_{crit} and a_p with δ for the three different time- delays $\tau_1 = 0.0$, $\tau_1 = \pi/4\omega_0$ and $\tau_1 = \pi/2\omega_0$. Here, as in the case of primary resonance, the optimal control performance can be achieved by the selection of $\tau_1 \prec \pi/\omega_0$ and $\delta \prec \pi/\omega_0$.

Fig. 7 shows the superharmonic frequency–response curves for three different sets of the time-delays. There exists a region of coexistence of the three solutions for $\tau_1 = 0.0$, $\delta = 0.0$ and $\tau_1 = \pi/\omega_0$, $\delta = 3\pi/4\omega_0$. The bending of the frequency–response curves is responsible for a jump phenomenon. The value of the detuning parameter σ for saddle-node bifurcation is larger for $\tau_1 = \pi/\omega_0$, $\delta = 3\pi/4\omega_0$ than that for $\tau_1 = 0.0$, $\delta = 0.0$. This indicates that the occurrence of saddle-node bifurcation and jump phenomenon can be delayed by certain values of the time-delays. For $\tau_1 = \pi/4\omega_0$ and $\delta = \pi/2\omega_0$, there is no jump and hysteresis phenomena. This again

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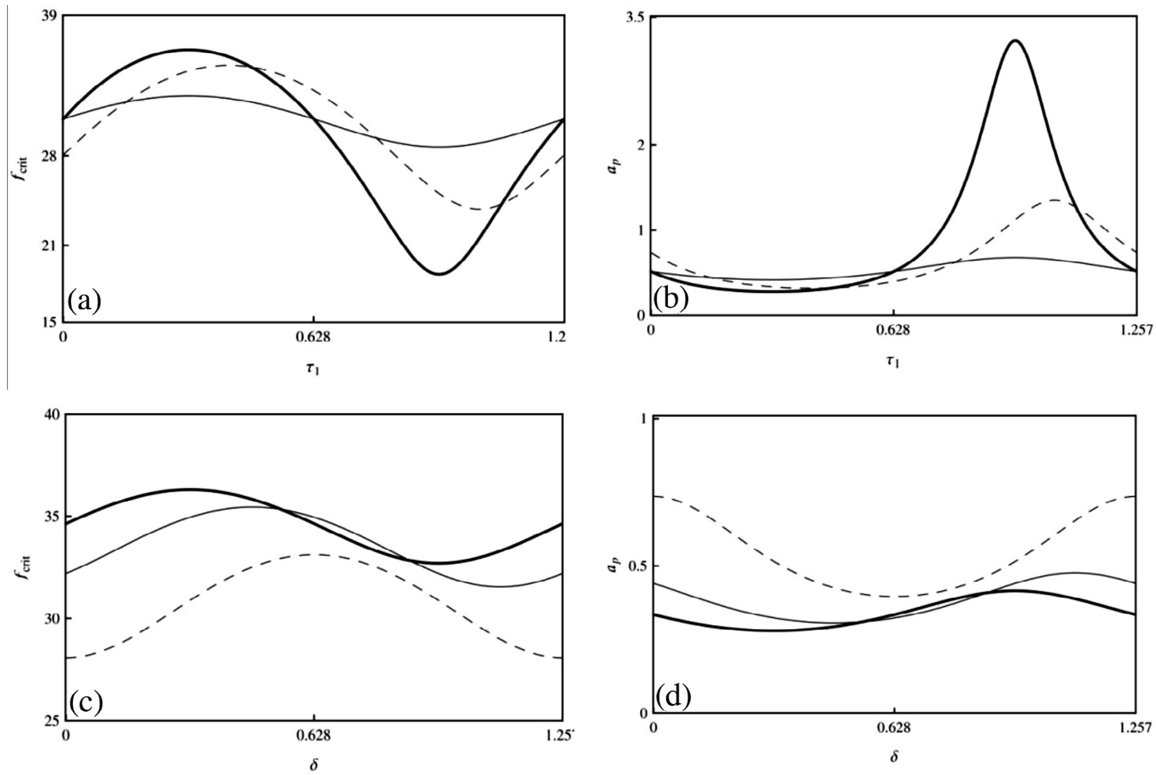

Figure 6 The critical force amplitude f_{crit} for the superharmonic resonance response of order 5 and the peak amplitude of the free oscillation term a_p as a function of the time-delays: (a) and (b) ----- curves are for $\delta = 0.0$, —— curves are for $\delta = 3\pi/2\omega_0$ and **——** curves are for $\delta = \pi/2\omega_0$; (c) and (d) ----- curves for $\tau_1 = 0.0$, —— curves for $\tau_1 = \pi/4\omega_0$ and **——** curves for $\tau_1 = \pi/2\omega_0$.

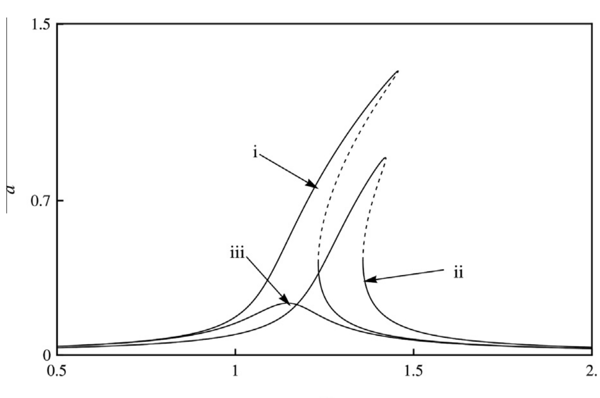

Figure 7 Superharmonic Frequency–response curves corresponding to three sets of the time-delays under $d_m = 0.27$ and $d_n = 0.03$: (i) are curves for $\tau_1 = 0.0$ and $\delta = 0.0$, (ii) are curves for $\tau_1 = \pi/\omega_0$ and $\delta = 3\pi/4\omega_0$ and (iii) are curves for $\tau_1 = \pi/4\omega_0$ and $\delta = \pi/2\omega_0$.

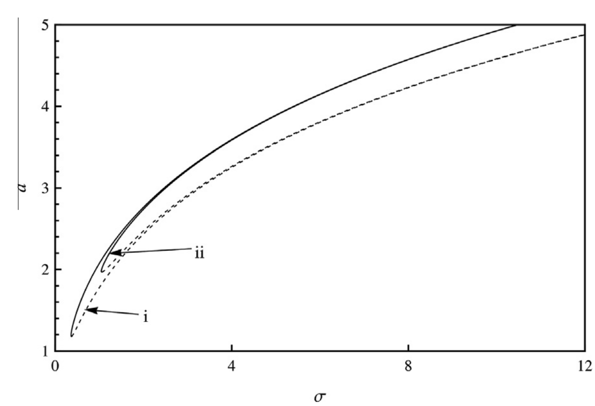

Figure 8 Subharmonic Frequency–response curves for the uncontrolled system (i) and the controlled system (ii) for ($\tau_1 = \pi/4\omega_0$, $\delta = \pi/4\omega_0$) corresponding to the excitation amplitude $f = 35.0$ and under the feedback gains $d_m = 0.135$ and $d_n = 0.015$.

suggests that saddle node bifurcation and jump phenomenon can be eliminated by certain values of the time-delays. Thus, the control performance can be enhanced by the optimal selection of the feedback gains and time-delays.

Frequency–response Eq. (29) is a nonlinear algebraic equation in the amplitude a. This equation and stability condition (30) are solved numerically by the Software of Mathematica. The numerical results are plotted in Fig. 8. This figure shows the frequency–response curves for subharmonic oscillations of order one-three. The frequency–response curves of non-trivial solutions are multi-valued and consist of two branches: one is stable and the other is unstable, also the branches of the response curves are bent to the right.

For the subharmonic resonance response, the time-delays can change the regime for the occurrence of subharmonic resonance. Fig. 9 shows the regions where subharmonic response exists for the three different sets of time-delays. It is noted that the regions for the existence of subharmonic responses are

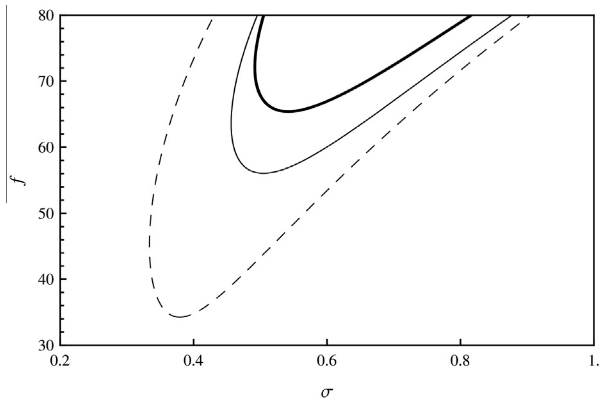

Figure 9 Regions where subharmonic responses exist for three sets of the time- delays under $d_m = 0.135$ and $d_n = 0.015$: ----- curves are for $\tau_1 = 0.0$ and $\delta = 0.0$, —— curves are for $\tau_1 = \pi/4\omega_0$ and $\delta = 3\pi/2\omega_0$, and **——** curves are for $\tau_1 = \pi/4\omega_0$ and $\delta = \pi/4\omega_0$.

different. When no time-delays are implemented in the feedback control (i.e., $\tau_1 = 0.0$, $\delta = 0.0$), the region is the largest one. For a fixed $\tau_1 = \pi/4\omega_0$, the region gets smaller at $\delta = 3\pi/2\omega_0$ and $\delta = \pi/4\omega_0$. When $\tau_1 = \pi/4\omega_0$ and $\delta = \pi/2\omega_0$, region of subharmonic resonance is reduced . This indicates that there always exist certain regimes of the time-delays where subharmonic response does not exist.

4. Conclusions

In this paper, we have presented an analysis of primary, superharmonic of order five, and subharmonic of order one-three resonances for non-linear s.d.o.f. system with two distinct time-delays under an external excitation. The method of multiple scales is used to determine two first order ordinary differential equations which describe the modulation of the amplitudes and the phases. Steady-state solutions and their stabilities in each resonance are investigated. Numerical results are obtained by using the Software of Mathematica, which presented in a group of figures. The effect of the feedback gains and time-delays on the non-linear response of the system is discussed and it is found that: an appropriate feedback can enhance the control performance. A suitable choice of the feedback gains and time-delays can enlarge the critical force amplitude, and reduce the peak amplitude of the response (or peak amplitude of the free oscillation term) for the case of primary resonance (superharmonic resonance). Furthermore, a proper feedback can eliminate saddle-node bifurcation, thereby eliminating jump and hysteresis phenomena taking place in the corresponding uncontrolled system. For subharmonic resonance, an adequate feedback can reduce the regions of subharmonic resonance response.

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